Math 307 - Differential Equations - Spring 2017
Quiz 10
April 27, 2017

Name: Solution

Problem 1. *Let* $f(x) = \begin{cases} 0, & -L < x < 0 \\ L, & 0 < x < L \end{cases}$. *Find the Fourier transform of* $f(x)$ *on the interval* $[-L, L]$. *For a bonus 3 points, use the Fourier series of this function to show that*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{\cancel{L}}\int_0^L \cancel{L}\,dx = L$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{\cancel{L}}\int_0^L \cancel{L}\cos\frac{n\pi x}{L}\,dx = \frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{\cancel{L}}\int_0^L \cancel{L}\sin\frac{n\pi x}{L}\,dx = \frac{-L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L$$

$$= \frac{-L}{n\pi}(\cos n\pi - 1) = \frac{L}{n\pi}\left(1+(-1)^{n+1}\right) = \begin{cases} 0, & n \text{ even} \\ \frac{2L}{n\pi}, & n \text{ odd} \end{cases}$$

$$F(x) = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{L}{n\pi}\left(1+(-1)^{n+1}\right)\sin\frac{n\pi x}{L} = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{L}$$

ExCr: $x = L/2$ $\quad F(L/2) = L = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L}{(2n-1)\pi}\sin\frac{(2n-1)\pi}{2} = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L(-1)^{n+1}}{(2n-1)\pi}$

Subtract $L/2$ $\quad \Rightarrow \frac{L}{2} = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = \frac{2L}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+2}}{2n+1} = \frac{2L}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$

Multiply by $\frac{\pi}{2L}$ $\quad \Rightarrow \frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$